Understanding of indices in some famous equations

Nguyen, Thanh **Luan**
Computational Engineering

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Vector & Tensor

- A vector is described as

$$\underline{u} = u_i \underline{e}_i$$
$$= u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3$$

Note on summation convention:

1. Each index runs over 1, 2, 3
2. Free index emerges only once
3. Dummy index emerges twice and twice only

- A 2nd order tensor is a result of dyadic product of 2 vectors

$$\underline{T} = \underline{u} \otimes \underline{v} = T_{ij} \underline{e}_i \otimes \underline{e}_j$$
$$T_{ij} = u_i v_j$$

- A tensor may have higher order, i.e. 3rd order, 4th order,…

Why Tensor?

- Materials in nature are deformable. A scalar or a vector cannot represent deformation. In order to describe and understand various modes of deformation a tensor is needed to represent **deformation state**.
- Inside a deforming material body, there emerges interacting contact area forces due to resistance of the material body to deformation. This contact force on each area element can be determined by applying a *transformation-like* quantity on the normal vector of the element area. This transformation-like quantity represents the **stress state** and is called **stress tensor**.

Some basic operations

- Kronecker delta notation

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} \qquad \delta_{ij} = \underline{e}_i . \underline{e}_j$$

$$\underline{H}_{\ldots i \ldots} \delta_{ij} = \underline{H}_{\ldots j \ldots}$$

- Dot-product of 2 vectors

$$\underline{u}.\underline{v} = (u_i \underline{e}_i).(v_j \underline{e}_j)$$

$$= u_i v_j \underline{e}_i . \underline{e}_j = u_i v_j \delta_{ij}$$

$$= u_i v_i$$

Rules of 1-dot product:

1. Find out 1 pair of neighboring basic vectors
2. Multiply this pair of vectors

Some basic operations

- Dot-product of a tensor and a vector

$$\underline{T}.\underline{u} = \left(T_{ij}\underline{e}_i \otimes \underline{e}_j\right).\left(u_k \underline{e}_k\right)$$

$$= T_{ij}u_k\left(\underline{e}_j.\underline{e}_k\right)\underline{e}_i$$

$$= T_{ij}u_k\delta_{jk}\underline{e}_i = T_{ij}u_j\underline{e}_i$$

<u>Rules of 1-dot product:</u>

1. Find out 1 pair of neighboring basic vectors
2. Multiply this pair of vectors

Some basic operations

- Dot-product between a 4th order tensor and a 2nd order tensor

$$\underline{C} : \underline{S} = \left(C_{ijkl} \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l \right) : \left(S_{ab} \underline{e}_a \otimes \underline{e}_b \right)$$

$$= C_{ijkl} S_{ab} \delta_{ka} \delta_{lb} \underline{e}_i \otimes \underline{e}_j$$

$$= C_{ijkl} S_{kl} \underline{e}_i \otimes \underline{e}_j$$

$$= T_{ij} \underline{e}_i \otimes \underline{e}_j$$

<u>Rules of 2-dot product:</u>

1. Find out 2 pairs of neighboring basic vectors
2. Multiply these 2 pairs by order

Gradient and Divergence

- Nabla operator

$$\nabla = \underline{e}_i \frac{\partial}{\partial x_i} = \underline{e}_i \partial_i$$
$$= \underline{e}_1 \partial_1 + \underline{e}_2 \partial_2 + \underline{e}_3 \partial_3$$

- Gradient of a scalar

$$\nabla f = \underline{e}_i \frac{\partial f}{\partial x_i} = \underline{e}_i \partial_i f = \underline{e}_i f_{,i}$$
$$= \underline{e}_1 f_{,1} + \underline{e}_2 f_{,2} + \underline{e}_3 f_{,3}$$

- Gradient of a vector

$$\nabla \underline{u} = \left(\underline{e}_i \partial_i\right)\left(u_j \underline{e}_j\right) = u_{j,i} \underline{e}_i \otimes \underline{e}_j$$

- Divergence of a vector

$$\nabla . \underline{u} = \left(\underline{e}_i \partial_i\right).\left(u_j \underline{e}_j\right) = u_{j,i} \underline{e}_i . \underline{e}_j = u_{i,i}$$
$$= u_{1,1} + u_{2,2} + u_{3,3}$$

Gradient and Divergence

- Divergence of a 2nd order tensor

$$\nabla . \underline{T} = \left(\underline{e}_a \partial_a\right) . \left(T_{ij} \underline{e}_i \otimes \underline{e}_j\right)$$

$$= \partial_a T_{ij} \delta_{ai} \underline{e}_j = \partial_i T_{ij} \underline{e}_j = T_{ij,i} \underline{e}_j$$

Application

Linear Elasticity	Groundwater Flow
Essential variable: Displacement **u**	Water pressure p
Kinematics:	
$\underline{\varepsilon} = \frac{1}{2}\left(\nabla\,\underline{u} + \underline{u}\nabla\right)$	$\underline{I} = -\nabla\,p$
$\underline{\varepsilon} = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right)\underline{e}_i \otimes \underline{e}_j$	$\underline{I} = -\underline{e}_i\,p_{,i}$
$\varepsilon_{ij} = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right)$	$I_i = -p_{,i}$

__Notation:__ ε – **linear strain tensor**, I – **pressure gradient vector**

Application

Linear Elasticity | **Groundwater Flow**

Constitutive: Hooke's law

$$\underline{\sigma} = \underline{E} : \underline{\varepsilon}$$

$$\underline{\sigma} = E_{ijkl} \varepsilon_{kl} \underline{e}_i \otimes \underline{e}_j$$

$$\sigma_{ij} = E_{ijkl} \varepsilon_{kl}$$

Darcy's law

$$\underline{q} = \underline{K} . \underline{I}$$

$$\underline{q} = K_{ij} . I_j \underline{e}_i$$

$$q_i = K_{ij} . I_j$$

Notation: E – 4th order **material tensor**, K – 2nd order **permeability tensor**
σ – 2nd order **stress tensor**, q – **Darcy's flux vector**

Application

Linear Elasticity | **Groundwater Flow**

Equilibrium: $\nabla . \underline{\sigma} + \underline{f} = 0$

$\nabla . \underline{q} = Q$

$\sigma_{ij,i} \underline{e}_j + f_j \underline{e}_j = 0$

$q_{i,i} = Q$

$\sigma_{ij,i} + f_j = 0$

Notation: f – **force vector**, Q – **Recharge**

The End

Thank you for listening!